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Best Approximation and Optimization

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In this expository paper it is shown how some of the main ideas of the theory of best approximation have been generalized to yield new methods and results in optimization theory and how they have continued to develop within the framework of optimization theory.

1

We recall that in a normed linear space (nls) F the *distance* from an element $x_0 \in F$ to a subset G of F is the number

$$\operatorname{dist}(x_0, G) = \inf_{g \in G} \|x_0 - g\|; \tag{1}$$

any $g_0 \in G$ for which this inf is attained, i.e., such that

$$\|x_0 - g_0\| = \inf_{g \in G} \|x_0 - g\|, \tag{2}$$

is called a (or an *element of*) *best approximation* of x_0 in G, or a *nearest point* to x_0 in G, and we shall denote the set of all such $g_0 \in G$ by $\mathscr{P}_G(x_0)$.

On the other hand, in optimization theory one is concerned, for a given locally convex space (lcs) F, a subset G of F and a functional $h: F \to \bar{R} = [-\infty, +\infty]$, with the number

$$\inf h(G) = \inf_{g \in G} h(g); \tag{3}$$

any $g_0 \in G$ for which this inf is attained, i.e., such that

$$h(g_0) = \inf h(G), \tag{4}$$

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is called a *solution* of the optimization problem (3), or of the program $(G, h|_G)$ and we shall denote the set of all such $g_0 \in G$ by $\mathscr{S}_G(h)$. We shall assume here, only for simplicity, that all spaces F are real.

Clearly, problem (1) is a particular case of problem (3), by taking, for the given element $x_0 \in F$, the functional

$$h(y) = \|x_0 - y\| \qquad (y \in F). \tag{5}$$

Hence the theory of best approximation (tba) may be regarded as a particular field of applications of optimization theory (ot). This fact has been observed in the 1960s, following the previous independent development of the two theories (for some references, see, e.g, the survey papers [38, 17]). In the 1970s, some books have been written in this spirit [19, 13, 16] and the same point of view has also appeared in parts of other monographs on ot. On the other hand, starting with [33], we have suggested a program of work in the opposite direction, i.e., to show that many methods and results of the tba are so strong that they can be generalized to yield new methods and results in ot. Subsequently, this point of view has been also adopted by others (see, e.g., [49, 4]). In the expository paper [38] we have presented some of our contributions to the interaction between the tba and ot, up to August 1979, with special emphasis on the above-mentioned program.

In the present paper we want to show how some of the main ideas of the tba have been generalized to yield new methods and results in ot and how they have been developed further within the framework of ot (often by authors extraneous to the tba). The paper is expository in nature, but it contains also some new remarks. The intersection of this paper with the survey paper [38] is minimal. Furthermore, in order to keep the presentation short, we have omitted some topics (e.g., minimizing sequences, connections with Hahn–Banach extensions, generalizations of moment problems to systems) and we have given only some samples of references. Nevertheless, we hope that the present paper will stimulate the interest of some of the specialists in the tba for this direction of research.

2

One of the oldest results in the tba is that if G is a finite-dimensional (linear) subspace of a nls F and $x_0 \in F$, then $\mathscr{S}_G(x_0) \neq \varnothing$ (see [31] for references). The proof is based on the fact that G is closed, each ball $B(x_0, c) = \{y \in F \mid \|x_0 - y\| \leqslant c\}$ is compact and h of (5) is continuous on F; for some generalizations, within the tba, see the references in [32]. Cheney and Goldstein [6] have extended this idea to ot, by showing, e.g., that if F is a reflexive Banach space, G a closed subset of F and $h: F \to R = (-\infty, +\infty)$ a functional, such that the level sets

$$S_c(h) = \{y \in F \mid h(y) \leqslant c\} \qquad (c \in R) \tag{6}$$

are closed (i.e., h is lower semi-continuous = lsc), convex (i.e., h is quasi-convex) and bounded (whence weakly compact), then $\mathscr{S}_G(h) \neq \varnothing$; more generally, one can replace (see, e.g., [10, p. 34]) the assumption that all $S_c(h)$ are bounded by the assumption that h is "coercive" on G, i.e., all $G \cap S_c(h)$ $(c \in R)$ are bounded. The observation that for h of (5) we have $S_c(h) = B(x_0, c)$ $(c \geqslant 0)$, i.e., that the "good" extensions of the balls $B(x_0, c)$ are the level sets $S_c(h)$, plays an important role also in some other extensions of the tba to ot, as we shall see below.

3

It has been known (see, e.g., [31, p. 90]) that if G is a finite-dimensional subspace in a nls F, every local minimizer $g_0 \in G$ of (5) on G (i.e., such that (2) holds for G replaced by $G \cap V(g_0)$, where $V(g_0)$ is a neighbourhood of g_0) is a global minimizer of (5) on G. Rockafellar [28, p. 264] has observed that this holds if $G \subset F$ and $h: F \to \bar{R}$ are convex. Some extensions and some related results, involving various connectedness properties of $G \cap S_c(h)$ $(c \in R)$, can be found, e.g., in [23, pp. 139–140; 25, Chap. IV; 24]. Let us also mention (see [1, pp. 173–175]) that every local minimizer of $h|_G$ is a global minimizer of $h|_G$ if and only if the "level set multifunction" $c \to G \cap S_c(h)$ is lsc on $\{c \in R \mid G \cap S_c(h) \neq \varnothing\}$.

4

Another old result of the tba says that if G is a (linear) subspace in a strictly convex nls F and $x_0 \in F$, then $\mathscr{P}_G(x_0)$ is either empty, or a singleton (see, e.g., [31]). Extending this to ot, it has been shown (see, e.g., [10, p. 34]) that if G is a convex subset of a lcs F and $h: F \to \bar{R}$ is strictly convex on G (i.e., $h(\lambda g_1 + (1-\lambda) g_2) < \lambda h(g_1) + (1-\lambda) h(g_2)$ for all $y_1, y_2 \in F$ and $0 < \lambda < 1$), then $\mathscr{S}_G(h)$ is either empty, or a singleton. More generally (see [25, Theorem 4.2.6]), the same conclusion holds if G is convex and h is "strictly connected" on G, i.e., for any $g_1, g_2 \in G$, $g_1 \neq g_2$, there exists a continuous function $p: [0, 1] \to G$ such that $p(0) = g_1$, $p(1) = g_2$ and $h(p(\lambda)) < \max\{h(g_1), h(g_2)\}$ $(0 < \lambda < 1)$.

5

An easy application of a corollary of the Hahn–Banach theorem yields (see [31, pp. 18–20]) that for $x_0 \in F \backslash \bar{G}$, with G a subspace of a nls F, $g_0 \in G$ satisfies $g_0 \in \mathscr{P}_G(x_0)$ if and only if there exists $\Psi \in F^*$ (the conjugate space of F) such that $\|\Psi\| = 1$, $\Psi(g_0 - x_0) = \|x_0 - g_0\|$, and $\Psi(g) = 0$ $(g \in G)$; for some generalizations, within the tba, see the references in [32]. These characterization theorems for the elements of $\mathscr{P}_G(x)$ show already the importance of the set

$$M_{g_0 - x_0} = \{\Psi \in F^* \mid \|\Psi\| = 1, \Psi(g_0 - x_0) = \|x_0 - g_0\|\}, \tag{7}$$

of all "maximal functionals" of $g_0 - x_0$. Since for h of (5) we have (see, e.g., [33, Lemma 4.1]) $M_{g_0 - x_0} = \partial h(g_0)$, the subdifferential of h at g_0, defined by

$$\partial h(g_0) = \{\Psi \in F^* \mid \Psi(y - y_0) \leqslant h(y) - h(y_0) \qquad (y \in F)\}, \tag{8}$$

the "good" generalization of $M_{g_0 - x_0}$ to ot is the set $\partial h(g_0)$. Thus, for example, the following characterization theorem of Pšeničnyĭ–Rockafellar (see, e.g., [13, pp. 30–31]) is an extension of the above characterization theorem to ot: For a convex subset G of a lcs F and a continuous convex $h: F \to R$, $g_0 \in G$ satisfies $g_0 \in \mathscr{S}_G(h)$ if and only if there exists $\Psi \in F^*$ such that $\Psi \in \partial h(g_0)$, $\Psi(g_0) \leqslant \Psi(g)$ $(g \in G)$. In [33] we have used systematically the idea of replacing $M_{g_0 - x_0}$ by $\partial h(g_0)$, to obtain new characterization theorems for the elements of $\mathscr{S}_G(h)$.

More generally, for any $\varepsilon \geqslant 0$, the elements $g_0 \in G$ satisfying

$$\|x_0 - g_0\| \leqslant \inf_{g \in G} \|x_0 - g\| + \varepsilon \tag{9}$$

are called (see, e.g., [31]) *elements of* ε*-approximation* of x_0 in G and the elements $g_0 \in G$ satisfying

$$h(g_0) \leqslant \inf h(G) + \varepsilon, \tag{10}$$

are called ε*-solutions* of problem (3) ($\varepsilon = 0$ is the preceding case). Clearly, one can extend the characterization of elements of ε-approximation, given in [31, p. 163] (which is similar to the above, but with $\Psi(g_0 - x_0) \geqslant \|x_0 - g_0\| - \varepsilon$), to a characterization of ε-solutions, replacing $\partial h(g_0)$ by the ε-subdifferential of h at g_0, defined [5] by

$$\partial_\varepsilon h(g_0) = \{\Psi \in F^* \mid \Psi(y - y_0) \leqslant h(y) - h(y_0) + \varepsilon \ (y \in F)\}; \tag{11}$$

this has been done by Strodiot–Nguyen–Heukemes [48]. It is an interesting phenomenon that while in the initial version of their paper (report 80/12, Univ. of Namur), Section 5 has been devoted to ε-approximation, in the final version [48] any mention of "approximation" has disappeared.

6

For a convex subset G of a nls F and $x_0 \in F \setminus \bar{G}$, we have the well known (see, e.g., [13, p. 62]) duality formula

$$\operatorname{dist}(x_0, G) = \max_{\Psi \in F^*, \|\Psi\| = 1} |\Psi(x_0) - \sup \Psi(G)|, \tag{12}$$

where max denotes a sup which is attained. The usefulness of (12) for applications (see [32]) is due to the fact that for various concrete spaces F the general form of continuous linear functionals $\Psi \in F^*$ is well known and simple. Using the formula of Ascoli (see [31, p. 24]) on $\operatorname{dist}(x, H)$, where H is a (closed) hyperplane in F, one obtains the geometric interpretation of (9),

$$\operatorname{dist}(x_0, G) = \max_{H \in \mathscr{H}_{G, x_0}} \operatorname{dist}(x_0, H), \tag{13}$$

where $\mathscr{H}_{G,x_0}$ denotes the collection of all hyperplanes in F which support the set G and which separate G and x_0. This has been extended to ot in [34] (see, also, [38, 42]), but we shall mention here another result. From (13) it follows easily that

$$\operatorname{dist}(x_0, G) = \max_{D \in \mathscr{D}, D \supset G} \operatorname{dist}(x_0, D), \tag{14}$$

where $\mathscr{D}$ denotes the collection of all closed half-spaces in F. This has been extended to ot by Laurent and Martinet [20], namely, if G is a convex subset of a lcs F and $h: F \to \bar{R}$ is convex and upper semicontinuous at some $g_0 \in G$ with $h(g_0) < +\infty$, then

$$\inf h(G) = \max_{D \in \mathscr{D}, D \supset G} \inf h(D); \tag{15}$$

in analytical form, (15) is equivalent [41] to

$$\inf h(G) = \max_{\Psi \in F^*} \inf_{y \in F, \Psi(y) \leqslant \sup \Psi(G)} h(y). \tag{16}$$

The above-mentioned results of [20, 34] give sufficient conditions on G and h for certain duality formulae (such as (15), (16)) to hold. Some *necessary and sufficient* conditions for these formulae and other duality formulae to hold, such as

$$\inf h(G) = \max_{\Psi \in F^*} \inf_{y \in F, \Psi(y) \in \Psi(G)} h(y), \tag{17}$$

or of types (15)–(17) with max replaced by sup, have been given in [41], in terms of "closed," "open," and "nice" separation (in the sense of Klee [15]) of G from the level sets (6) and $A_c(h) = \{y \in F \mid h(y) < c\}$ $(c \in R)$; let us mention that a generalization of closed separation [15] has been used by Rubinštein [29, 30], who has constructed a different general scheme of obtaining duality theorems for optimization problems. The methods of [41] have been further developed in [42] for the case $\inf h(F) < \inf h(G)$ (which, for h of (5), is equivalent to $x_0 \in F \backslash \bar{G}$, a natural assumption in the tba), in [43] for characterizations of solutions $g_0 \in G$ of (3) and in [44] to the case of "surrogate duality" (generalizing [11, 12]), which aims, roughly speaking, at expressing $\inf h(G)$ with the aid of $\inf h(M)$, with suitable sets M related to G, such as in Eqs. (15)–(17); a more flexible general theory of surrogate duality, which encompasses some classical particular cases, involving two spaces F, X (instead of one space F), has been given in [47]. Thus, we see that it is actually surrogate duality, rather than Lagrangian duality, which is used in the tba, and that the methods of the tba, extended to ot, have led to a richer theory of surrogate duality. For a comparison with Lagrangian duality, see [35, 38, 39].

Let us also mention that the similar extension to ot, of some results on best approximation by "caverns" G in a nls F, has led to the discovery of some new separation theorems for bounded convex sets in normed linear spaces [36, 37], which may have interest also for other applications.

7

There are, in the tba, some basic concepts involving best approximation by G of more than one $x \in F$. For example, a set G in a nls F is called (a) proximinal, if $\mathscr{P}_G(x) \neq \varnothing$ $(x \in F)$; (b) semi-Čebyšev, if $\mathscr{P}_G(x) = \varnothing$ or singleton $(x \in F)$; (c) Čebyšev, if $\mathscr{P}_G(x) =$ singleton $(x \in F)$; (d) almost Čebyšev, if $\{x \in F \mid \mathscr{P}_G(x) = \text{singleton}\}$ is at most of the first category in F, etc. Furthermore, the idea of considering, istead of problem (1) for one fixed $x_0 \in F$, a family of problems of type (1), for several $x \in F$, occurs naturally also when considering properties of the best approximation operator $x \to \mathscr{P}_G(x)$ (set-valued or, when G is a Čebyšev set, single-valued), e.g., semicontinuity, continuity, etc.

A natural and important extension to ot, due to Rockafellar [27] (see also [14]), is that of *parametrization* (or *perturbation*) of the optimization problem (3); although Rockafellar has not stated explicitly that he arrived at it from the model of the tba, he had a very good knowledge of this model (see, e.g., [18]). Simplifying things, parametrization of (3) amounts to considering a set of parameters, say X, and a functional $\varphi: F \times X \to \bar{R}$, such that for some $x_0 \in X$ there holds

$$h(y) = \varphi(y, x_0) \qquad (y \in F), \tag{18}$$

so (3) becomes the problem

$$\inf \varphi(G, x_0) = \inf_{g \in G} \varphi(g, x_0). \tag{19}$$

In this way, problem (3) is "embedded" into the one-parameter family of optimization problems

$$f(x) = \inf_{g \in G} \varphi(g, x) \qquad (x \in X); \tag{20}$$

for example, in the tba, taking $X = F$ and $\varphi(y, x) = \|x - y\|$ for all $x, y \in F$, condition (18) is satisfied by h of (5) and we have $f(x) = \operatorname{dist}(x, G)$ $(x \in X)$.

This method of parametrization, combined with various concepts of conjugation of functionals, has permitted to define general concepts of dual problems to (3) and it has turned out that the properties of duality (e.g., existence of solutions of the dual problem) are equivalent to properties of "stability" of problem (19), with respect to small perturbations of the parameter x_0, i.e., to properties (when X is lcs) of the "optimal value functional" f of (20) at x_0. In the particular case of the tba, f has "good" properties (it is finite, convex, and continuous on $X = F$) and therefore so do the dual problems, but for the study of more general problems (3) on a lsc F, the above-mentioned connections between duality and stability are revealing.

It is clear how the concepts of the tba, involving best approximation by G of several $x \in F$, extend to ot, with the above method of embedding (e.g., to proximinality there corresponds the property that for each $x \in X$, problem (20) has a solution $g_x \in G$, etc.). In this direction, let us mention the results of Baranger–Temam [2] and Lebourg [21] on the existence of solutions $g_x \in G$ of problem (20) for x ranging in a dense (or in a dense G_δ) subset of X, which generalize known results of the tba. Moreover, this scheme encompasses also extensions to ot of results on farthest points and of other results of Banach space theory, e.g., of the Bishop–Phelps theorem (see [5, 21, 9]).

8

One can extend the best approximation operator $x \to \mathscr{P}_G(x)$ to ot, with the aid of the above parametrization, defining the "optimization operator" $x \to \mathscr{S}_G(\varphi_x)$, where, for each $x \in X$, $\varphi_x: F \to \bar{R}$ is the "partial functional" defined by $\varphi_x(y) = \varphi(y, x)$ $(y \in F)$ (in particular, by (18), $\varphi_{x_0} = h$).

Assuming that X is a topological (usually, a locally convex) space, one can study, e.g., the semi-continuity of this operator. Another "optimization operator" has been also considered, namely, $h \to \mathscr{S}_G(h)$, defined on $\bar{R}^F$ endowed with various concepts of convergence (see, e.g., [51]). However, let us observe that this is encompassed by the above scheme, by taking $X = \bar{R}^F$ and $\varphi(y, h) = h(y)$ for all $y \in F$, $h \in \bar{R}^F$; indeed, then $\varphi_h = h$, for all $h \in \bar{R}^F$.

On the other hand, in the tba, the operator $G \to \mathscr{P}_G(x_0)$ from 2^F into 2^G, with an arbitrary fixed $x_0 \in F$, has been also studied (see, e.g., [22]). The obvious extension to ot is the operator $G \to \mathscr{S}_G(h)$, with a fixed $h: F \to \bar{R}$ (see, e.g., [4]). This method of keeping h fixed and letting G vary in 2^F, can be also applied to other problems of ot, e.g., Wriedt [50] has proved that if F is a lcs and $h: F \to R$ is continuous convex, with $\mathscr{S}_F(h)$ bounded, the following statements are equivalent: (1) For each nonempty closed convex set $G \subset F$, we have $\mathscr{S}_G(h) \neq \varnothing$; (2) F is a reflexive Banach space and all level sets (6) are bounded. Also, Wriedt has proved [49, Theorem 1] that if $h: F \to \bar{R}$ is quasi-convex, the following statements are equivalent: (1) For each convex set $G \subset F$, $\mathscr{S}_G(h) = \varnothing$ or singleton. (2) For each segment $G \subset F$, $\mathscr{S}_G(h) = \varnothing$ or singleton. (3) h is "strictly quasi-convex" (in the sense that the relations $y_1, y_2 \in F$, $y_1 \neq y_2$, $h(y_1) = h(y_2) = r$, $0 < \lambda < 1$, imply $h(\lambda y_1 + (1 - \lambda) y_2) < r$). In these situations, G is perturbed by subsets G' of F. It is also convenient to consider a one-parameter family $\{\Gamma(x)\}_{x \in X}$ of perturbation subsets $\Gamma(x) \subset F$, where X is a topological space and $\Gamma: X \to 2^F$ is a multifunction such that $\Gamma(x_0) = G$ for some $x_0 \in X$, so (3) is "embedded" into the family of problems

$$f(x) = \inf_{y \in \Gamma(x)} h(y) \qquad (x \in X), \tag{21}$$

for which one can again define dual problems and study duality-stability relations (see, e.g., [8, 45]), properties of the "optimization operator" $x \to \mathscr{S}_{\Gamma(x)}(h)$ [3], etc. Let us observe that this method encompasses the preceding case, in which G varies in 2^F, by taking $X = 2^F$, $x_0 = G$, and $\Gamma(x) = x$ for all $x \in X$. On the other hand, as has been observed in [40], the scheme of Rockafellar, with $\varphi: F \times X \to \bar{R}$, encompasses the scheme with a multifunction $\Gamma: X \to 2^F$, provided we replace in (3), (18), h by $h + \chi_G$, where $\chi_G(y) = 0$ for $y \in G$ and $= +\infty$ for $y \notin G$ (which does not alter problems (3), (19)), and we replace (20) by $f(x) = \inf_{y \in F} \varphi(y, x)$ $(x \in X)$; indeed, then, for $\Gamma: X \to 2^F$ as above, defining $\varphi(y, x) = h(y) + \chi_{\Gamma(x)}(y)$ for all $y \in F$, $x \in X$, the modified (20) reduces to (21). This method, applied to the particular case when $X = F$, $x_0 = 0$ and,

$$\Gamma(x) = G - x \qquad (x \in F) \tag{22}$$

and combined with various concepts of conjugation of functional, yields also problems (16), (17) and other similar problems as particular cases of the general concepts of dual problems mentioned in Section 7 (see [46]).

In the particular case when $X=F$, $x_0=0$, and Γ is defined by (22), we have

$$f(x)=\inf_{g\in G} h(g-x) \qquad (x\in F), \tag{23}$$

which, for h of (5) (with $x_0=0$), yields $f(x)=\operatorname{dist}(x, G)$ $(x\in X)$, the case of the tba; see, e.g., [49] for results on the "optimization operator" $x\to \mathscr{S}_{G-x}(h)=\{g_0\in G \mid h(g_0-x)=\inf_{g\in G} h(g-x)\}-x$, for convex G, h.

In the tba, the case when both $x\in F$ and $G\in 2^F$ vary, is also often considered, e.g., in the problem of characterization of the spaces F such that each convex set $G\subset F$ is proximinal, or Čebyšev, etc. The obvious extension to ot is the case when both h and G vary, or, in one-parametric version, the embedding of (3) into the family of optimization problems

$$f(x)=\inf_{y\in\Gamma(x)} \varphi(y,x) \qquad (x\in X), \tag{24}$$

where φ and Γ are as above. The properties of the "optimization operators" $(G,h)\to\mathscr{S}_G(h)$ and $x\to\mathscr{S}_{\Gamma(x)}(\varphi_x)$ have been studied, e.g., in [7, 26, 51], etc. Let us observe that, replacing the space F by $F\times X$ and defining $\Delta: X\to 2^{F\times X}$ by $\Delta(x)=(\Gamma(x), x)$ for all $x\in X$, the relation $y\in\Gamma(x)$ holds if and only if $(y,x)\in\Delta(x)$, so (24) becomes $f(x)=\inf_{(y,x)\in\Delta(x)}\varphi(y,x)$, i.e., of type (21), for problem (19); also, by $\Gamma(x_0)=G$, we have $\Delta(x_0)=(\Gamma(x_0), x_0)=(G, x_0)$.

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